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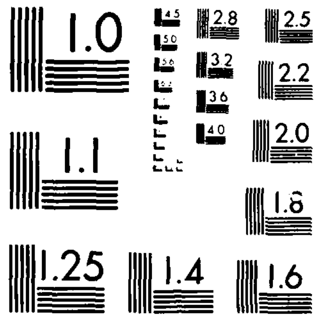


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# THE MATHEMATICAL STRUCTURE OF

# ERROR CORRECTION MODELS

by

Søren Johansen*



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*University of Copenhagen



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## 0. Abstract.

The error correction model for a vector valued time series has been proposed and applied in the economic literature with the papers by Sargan(1964), Davidson et al.(1978), Hendry and von Ungern-Sternberg(1981) and has been given a formal mathematical treatment by Granger(1983). He introduced the notion of cointegratedness of a vector process and showed the relation between cointegration and error correction models.

This paper defines a general error correction model, that encompasses the usual error correction model as well as the integral correction model by allowing a finite number of error correction terms which correspond to linear combinations of the vector process that are integrated of different order.

It is shown that this structure is inherent in the model if it is given in autoregressive form or moving average form by exploiting the singularity of the matrix function that defines the model.

The theory is applied to some examples discussed by Davidson(1983) and Harvey(1982).

Key words: Cointegration, error correction, non-stationary time series, ARMA models.

## 1. Introduction.

Many of the current controversies concerning macroeconomic policy issues, particularly monetary questions, seem to derive from different views of the duration and importance of short-run and long-run behavior of economic agents. The fact that these controversies have to be resolved in the empirical arena, can clearly be demonstrated by a growing number of empirical applications in which the dynamics of short-run and long-run adjustment processes are being modelled. In particular, the idea of incorporating the dynamic adjustment to long-run steady-state targets in the form of error correcting mechanisms, in an autoregressive model for transitory short-run dynamic behavior, originally suggested by Sargan(1964) and further developed by Davidson et al.(1978), Hendry and von Ungern-Sternberg(1981), Davidson and Hendry(1981),etc. seems to have introduced a useful approach to modelling the dynamics of economic behavior.

The error correction model is a model that combines the autoregressive form for the changes in $y_t$ with existing economic theory as expressed in the long-term static relation

$$Ay + Bz = 0 \tag{1.1}$$

or in the steady-state growth relation

$$Ay + B(z + \Delta z) = 0. \tag{1.2}$$

An example of an error correction model is given by

$$A_1(L)\Delta y_t + B_1(L)\Delta z_t + D(L)[Ay_{t-1} + Bz_{t-1}] = C(L)\epsilon_t. \tag{1.3}$$

Applications of this type of models usually have proved very successful in terms of model fit, meaningful estimates of parameters of interest, encompassing,etc. However, for a long time a formal mathematical treatment

seemed to be lacking. Granger(1983) was the first to provide such a basis by introducing the concept of cointegratedness between time series and relating that to the concept of error correcting models. The idea of Granger, see also Granger and Weiss(1983) and Granger and Engle(1985), is to start with a general model for $x_t = \{y_t, z_t\}$ expressing that $\Delta x_t$ is stationary, i.e.

$$\Delta x_t = C(L)\epsilon_t \tag{1.4}$$

and then showing that certain properties of the matrix function C(L) imply that the components of $x_t$ are cointegrated. This is then used to derive an error correction model. The purpose of this is to explain combinations of the $\epsilon$'s in terms of deviations in long-term relations between the non-stationary components of the vector process $\{x_t\}$, and to identify these long-term relations using the concept of cointegration.

The purpose of this paper is to discuss the equations (1.1)-(1.4) and some related concepts from a mathematical point of view, find their interrelations, and provide a framework in which their formal analysis can be justified.

In doing so the concepts are clarified and generalised. We thus end up with a very general type of model for a class of non-stationary stochastic processes. Not all these models correspond to interesting economic models but their structure permits a simple analysis and thus helps the understanding of the interesting examples which are discussed in the economic literature.

The paper is now organised as follows :

Section 2 discusses the basic properties of some non-stationary processes. In particular the role of the starting values and their influence on the process is discussed.

Following Granger(1983) Section 3 discusses the problem of determining

the long-range relations in a system given by the moving average representation

$$\Delta^d x_t = C(L)\epsilon_t,$$

where d is the order of the process.We derive a general form of the error correction model, that allows error correction terms of different order. The usual error correction models, see Davidson et al.(1978), as well as the integral correction models, see Hendry and von Ungern Sternberg(1981) and Davidson(1983), can be seen as special cases of the general model. Section 4 discusses formally the same problem, but now based on the autoregressive representation

$$A(L)x_t = \Delta^p \epsilon_t,$$

where p is usually zero.

Conditions for this model to be interpreted as an error correction model are formulated, and the order of the process is found. In section 5 we discuss the special situation where $x_t$ is decomposed into endogeneous and exogeneous variables and derive an error correction model for the targetting error.

In section 6 we show how some examples from the economic literature can be treated by the general methods developed in the previous chapters, and Section 7 contains the mathematical results which consist of finding a representation of the determinant of a matrix valued function in terms of certain indices defined by the null spaces of succesive derivatives of the function at L = 1

## 2. The basic properties of a class of non-stationary processes.

It is customary to consider non-stationary processes $\{x_t\}$ given by the equation

(2.1) $$\Delta^d x_t = C(L)\epsilon_t$$

where $x_t$ and $\epsilon_t \in R^m$, and $C(z)$ is an $m$x$m$ matrix valued holomorphic function given by its power series with radius of convergence $1 + \rho$, $\rho > 0$, and d is an integer. Here $\{\epsilon_t\}$ is a sequence of independent identically distributed random variables with mean 0 and variance matrix $\Gamma$. Note that the coefficients of $C(z)$ decrease exponentially fast, which shows that $C(L)\epsilon_t$ is a stationary process. It is easy to construct $x_t$ recursively starting with $t = 0$, say. This is done as follows: We sum (2.1) from $t = 0$ to $t = T$ and find

$$\Delta^{d-1}x_T = \Delta^{d-1}x_{-1} + C(L)\sum_{t=0}^{T} \epsilon_t.$$

Summing again gives

$$\Delta^{d-2}x_t = (t+1)\Delta^{d-1}x_{-1} + \Delta^{d-2}x_{-1} + C(L) \sum_{0\le u\le s\le t} \epsilon_u.$$

It is seen that the process $x_t$ will be composed of two parts. The first is a polynomial of degree d-1 with coefficients depending on the past values of $x_t$, i.e. for $t < 0$. The second term is a repeated sum of the $\epsilon$'s. We can write

$$x_t = C(L)\Delta^{-d}\epsilon_t + P_{d-1}(t).$$

We call $P_{d-1}(t)$ the completely deterministic part of $x_t$ and $C(L)\Delta^{-d}\epsilon_t$ the random part of $x_t$. Note that $\Delta^{-1}$ is defined as a finite sum from 0 to t. The completely deterministic part can be considered a trend in the system showing the influence of the past, whereas the random part contains the cumulative effect of the shocks to the system. As a simple example consider the equation

$$\Delta^2 x_t = \epsilon_t, \ t \ge 0.$$

The solution is

$$x_t = x_{-1} + (t+1)\Delta x_{-1} + \sum_{0\le s\le u\le t}^{t} \epsilon_s .$$

Note that the trend part is just a straight line through points $\{-2,x_{-2}\}$ and $\{-1,x_{-1}\}$ and that conditionally on these values $x_t$ fluctuates around this

trend with a variance given by

$$V(x_t \mid x_{-1}, x_{-2}) = \sum_{s=0}^{t} (t-s+1)^2 V(\epsilon_1).$$

Hence the variance increases to infinity with t. This is often expressed by saying that the process has infinite variance. A stationary process can be started at "minus infinity" but a non-stationary process must be started at a finite time point, and the whole process has to be considered conditionally on the values before this time, otherwise the process is simply not defined by the equation (2.1). One can subtract $P_{d-1}(t)$ from $x_t$, since the difference also satisfies the differential equation, but now with the starting values zero.

This problem has implications for some of the formal calculations often applied to time series. Consider for instance the equation

$$\Delta^d x_t = C(L)\Delta^b \epsilon_t \ , \ t \geq 0.$$

If we sum b times we obtain

$$\Delta^{d-b} x_t = C(L)\epsilon_t + P_{b-1}(t),$$

where $P_{b-1}(t)$ has coefficients depending on the values of $\{x_t, \epsilon_t\}$ with $t < 0$. Thus one can cancel $\Delta^b$, at the expense of adding a trend of order b-1. One can also justify the cancellation of $\Delta^b$ as an operation on the random part of the process. In the following we shall in some examples be explicit about the trend, but the later examples only the random part will be dealt with in detail.

Next we turn to the notion of cointegration, see Granger(1983).

Definition.2.1. We shall call $x_t$ integrated of order d if $x_t$ has the representation

$$\Delta^d x_t = C(L)\epsilon_t + P_b(t)$$

where $P_b(t)$ is a completely deterministic polynomial of degree b, and $C(1) \neq$

0. In other words $x_t$ is integrated of order d if $\Delta^d x_t$ is stationary apart from a completely deterministic component. Notice that $C(1) \neq 0$ implies that $\Delta^{d-1} x_t$ is not stationary.

Definition 2.2. Let $x_t$ be integrated of order d. We shall call $x_t$ cointegrated with cointegration vector $\alpha \in R^m$ of order s if $\alpha' x_t$ is integrated of order d-s.

Thus the order of $x_t$ is reduced by s if the combination $\alpha' x_t$ is considered.

It is mathematically convenient to allow any vector $\alpha$ in the definition of cointegration. Thus if $\alpha = (1,0,0)'$ say, then $\alpha' x_t = x_{1t}$. We thus express the fact that some components of $x_t$ is in fact integrated of lower order than the whole vector process, by saying that a certain unit vector is a cointegration factor. This is clearly a slight abuse of the idea behind cointegration but it makes the formulation simpler.

3. The error correction model derived from the moving average representation.

We shall consider the equation

$$\Delta^d x_t = C(L)\epsilon_t \ , \ t \geq 0, \tag{3.1}$$

and assume that $C(z)$ is holomorphic for $|z| < 1+\rho$ , and is non-singular for $|z| \leq 1+\rho$ except for $z = 1$, where we assume that $C(1)$ is singular but $\neq 0$, since if $C(1)$ were 0 then a similar model would hold with d replaced by d-1. Note that we do not assume that each af the components of $x_t$ are integrated of the same order. Such an assumption is not necessary for the results developed below, but in connection with the examples this point will be discussed in more detail. We want to derive an error correction model for $x_t$, following the ideas of Granger(1983). We shall first give a general definition of an error correction model and then give some examples before we prove the main result.

The ultimate goal of this investigation is to be able to find the properties of a vector process $x_t$ from the defining equation in the autoregressive form

$$A(L)x_t = \epsilon_t \ , \ t \geq 0.$$

The equation, which defines the process uniquely, must therefore contain information on the order of the process and of which components are cointegrated. The problem is how to extract this information. If we can find out, that the order of integration is 2, say, then we can write the equation in the form

$$A_0 x_t + \Delta A_1 x_t + A_2(L)\Delta^2 x_t = \epsilon_t$$

and we want to interpret $A_0 x_t$ and $\Delta A_1 x_t$ as stationary error correction terms. Thus in order to interpret this as an error correction model we must make sure

that all terms represent stationary terms and that they have not been differenced too much. This is made precise in

Definition 3.1 A model of the form

$$\sum_{i=-s}^{k-1} \Delta^i D_i x_t + A(L)\Delta^k x_t = f(L)\Delta^d \epsilon_t \ , t \geq 0$$

is called a general error correction model of order k if

(3.2) $A(z)$ is holomorphic for $|z| < 1+\rho$ and $A(1) \neq 0$

(3.3) $\Delta^i D_i x_t$ is stationary, $i = -s,\dots,k-1$

(3.4) $x_t$ is integrated of order k

(3.5) $f(z) \neq 0 \ , \ |z| < 1+\rho.$

The terms $\Delta^i D_i x_t$, $-s \leq i < k$ represent error correction terms with cointegration factors $E_i$ of order ($\geq$) $k-i$, and the term $A(L)\Delta^k x_t$ gives the autoregressive model for the stationary process $\Delta^k x_t$. In general $d = 0$ but in some cases we need a different value. Note that if $i < 0$ then the term $\Delta^i D_i x_t$ is an integral correction term, i.e. the $|i|$ fold summation from 0 to t of $D_i x_t$.

Some examples will be given below

Example 3.1. Consider the process $(c_t, y_t)$ given by the equations

$$\Delta c_t = \beta \Delta y_t + \gamma(y_{t-1} - c_{t-1}) + \epsilon_{1t}$$

$$\Delta y_t = \epsilon_{2t}$$

This example has been adapted from example 6.1 and is treated in more detail in Section 6. It is easily seen that from the second equation it follows, that $y_t$ is integrated of order 1. Now the first equation only makes sense if $c_t$ is integrated of order 1 as well, in such a way that $y_t - c_t$ is integrated of order 0, since if for instance $c_t$ were integrated of order 2, then the left hand side would be of order 1 and the right hand side of order 2. Thus $y_t$ and $c_t$ are cointegrated with cointegration vector $\alpha' = (1,-1)$ of order 1, and the

model for $\Delta c_t$ is given in part by the autoregressive term $\beta\Delta y_t$ and in part by the stationary error correction term $y_{t-1}-c_{t-1}$.

Thus one can in a simple fashion identify the order of the vector process $(c_t, y_t)$ as well as the autoregressive part and error correction part in this example. Note that if the second equation is replaced by $\Delta^2 y_t = \epsilon_{2t}$, then the analysis changes and $c_t$ and $y_t$ become integrated of order 2 and $(y_t-c_t)$ becomes integrated of order 1. In this case the first equation should be multiplied by $\Delta$ before one can identify the autoregressive part and the error correction part, and then the error correction part is not a linear combination of the components of the process, but a linear combination of the components of the differenced process.

Example 3.2. A modified version of the previous example is given by the equation

$$\Delta c_t = \beta\Delta y_t + \partial\Delta^{-1}(y_{t-1}-c_{t-1}) + \epsilon_{1t}$$
$$\Delta y_t = \epsilon_{2t}$$

where $\Delta^{-1} z_t = \sum_{s=0}^{t} z_s$. A full treatement of a similar example is given in Section 6. At this point we shall use it to indicate that it may not be so obvious to find the order of the process $(c_t, y_t)$ and find out in what sense these equations determine an integral correction model. We shall only note that in general the error correction terms are stationary terms which are linear combinations of the vector process differenced a suitable number of times. Finally one can combine the two examples and consider an equation where both the error correction term as well as the integral correction term appear. Thus in general one can have many error correction terms in an error correction model corresponding to linear combinations which are integrated of different orders.

In order to state the main result about the error correction model we have to define three indices determined by the function $C(z)$. The necessary mathematical results are given in Section 7, but here we shall briefly recapitulate the definitions and results. From the assumptions on $C(z)$ it follows that we can expand it around $z = 1$ in a power series $C(z) = \sum_{j=0}^{\infty} (1-z)^j C_j$ which is convergent for $|z-1| < \rho$. The results of this section are formulated in terms of the coefficients $\{C_j\}$. We now let $N_j = \{x \in R^m | x'C_j = 0\}$, i.e. the null space for $C_j$. We then define the spaces $M_j = N_0 \cap \ldots \cap N_j$ of vectors which are null vectors for all matrices $C_i$, $i = 0,\ldots,j$. Let $m_j$ denote the dimension of $M_j$. Clearly the spaces $M_j$ are decreasing and since $C(z)$ is assumed to be regular for $z \neq 1$ there is no vector $x$ which is contained in all $N_j$. Hence there exists a $k$ such that

$$m > m_0 \geq \ldots \geq m_{k-1} > m_k = m_{k+1} = \ldots = 0.$$

Now define $n = \sum_{j=0}^{\infty} m_j = \sum_{j=0}^{k} m_j$ and let $r$ be defined by $\det C(z) = (1-z)^r f(z)$, where $f(z) \neq 0$.

Thus we associate with $C(z)$ the three numbers $(k,n,r)$ which will be used repeatedly in the following. We define $C_n(z)$ by the relation

$$C(z) = \sum_{j=0}^{n-1} (1-z)^j C_j + (1-z)^n C_n(z), \quad |z| < 1+\rho,$$

and the adjoint $\bar{C}(z)$ by

$$\bar{C}(z)_{ij} = (-1)^{i+j} \det C^{ji}(z)$$

where $C^{ji}(z)$ is found by deleting row $j$ and column $i$ from $C(z)$. Let

$$\bar{C}(z) = \sum_{j=0}^{n-1} (1-z)^j \bar{C}_j + (1-z)^n \bar{C}_n(z), \quad |z| < 1+\rho.$$

We can then formulate

Theorem 3.1. The process $x_t$ given by (3.1) satisfies an autoregressive

model of the form

$$\sum_{j=1}^{k} \Delta^{d-j}\bar{C}_{n-j}x_t + \bar{C}_n(L)\Delta^d x_t = f(L)\Delta^{r-n}\epsilon_t \ , \ t \geq 0. \tag{3.5}$$

If either $\bar{C}_n = \bar{C}_n(1) \neq 0$ or if $r = n$ this is a general error correction model of order d.

Proof. Theorem 7.3 shows that we have the following representation

$$\bar{C}(L) = \Delta^{n-k}\bar{C}_{n-k}(L) = \Delta^{n-k}\left[\sum_{j=0}^{k-1} \Delta^j\bar{C}_{n-k+j} + \Delta^k\bar{C}_n(L)\right] \tag{3.6}$$

and

$$\bar{C}_{n-j}C(L) = \Delta^j\bar{C}_{n-j}C_j(L) \ , \ j = 1,\dots,k \tag{3.7}$$

where $C_j(z)$ and $\bar{C}_n(z)$ are holomorphic in $|z| < 1+\rho$. Now the equation (3.1) defines $\Delta^d x_t$ as a stationary process, hence $\bar{C}(L)\Delta^d x_t$ is well defined and stationary and

$$\bar{C}(L)\Delta^d x_t = \bar{C}(L)C(L)\epsilon_t = f(L)\Delta^r\epsilon_t$$

where, by Theorem 7.1, $f(z) \neq 0$, $|z| < 1+\rho$. Now use the representation (3.6) and we get

$$\left[\sum_{j=n-k}^{n-1} \Delta^j\bar{C}_j + \Delta^n\bar{C}_n(L)\right]\Delta^d x_t = f(L)\Delta^r\epsilon_t$$

which implies that

$$\sum_{j=1}^{k} \Delta^{d-j}\bar{C}_{n-j}x_t + \bar{C}_n(L)\Delta^d x_t = f(L)\Delta^{r-n}\epsilon_t. \tag{3.8}$$

For this equation to be an error correction model we have to check the conditions (3.2)-(3.5). We know, since $C(1) \neq 0$, that $x_t$ is integrated of order d which shows (3.4). By multiplying (3.1) by $\bar{C}_{n-j}$ and using (3.7) we get

$$\bar{C}_{n-j}\Delta^{d-j}x_t = \bar{C}_{n-j}C_j(L)\epsilon_t \ , \ j = 1,\dots,k, \tag{3.9}$$

which shows that $\Delta^{d-j}\bar{C}_{n-j}x_t$ is stationary and hence that (3.3) is satisfied. Since $\bar{C}_n(z)$ is holomorphic and $f(z) \neq 0$, $|z| < 1+\rho$ we only have to check that $\bar{C}_n(1) \neq 0$. Thus if $\bar{C}_n = \bar{C}_n(1) \neq 0$ , then (3.8) is an error correction model, and if instead we assume that $r = n$, then it follows from Theorem 7.4 that $\sum_{j=0}^{k} \bar{C}_{n-j}C_j$ has rank m. Now since $C_0 = C(1) \neq 0$ we have $M_0^{\perp} \neq \{0\}$, and hence there exists $\lambda \neq 0$, $\lambda \in M_0^{\perp}$. Since the matrices $\{\bar{C}_{n-1},\ldots,\bar{C}_{n-k}\}$ span $M_0$ we find that

$$0 \neq \lambda' \sum_{j=0}^{k} \bar{C}_{n-j}C_j = \lambda'\bar{C}_n C_0$$

which implies that $\bar{C}_n \neq 0$ and the Theorem is proved.

The first term in the error correction model is

$$\Delta^{d-k}\bar{C}_{n-k}x_t$$

which will represent an integral correction term if $d < k$.

It is seen that if $r > n$, then a difference $\Delta^{r-n}$ remains at the $\epsilon$'s. This has the effect that when solving for $x_t$, one has to sum the error correction terms as well as the term $\Delta^{r-n}\epsilon_t$. The last one will contribute less to the variance of the process than the first one, and in this sense the main contribution to the variance of the process comes from the error correction terms. Thus one has in fact succeded in explaining the major part of the $\epsilon$'s in terms of interpretable error correction terms. If $r = n$ there is a balance between the two kinds of errors, and we therefore call this the balanced case. We shall see in the examples that both situations can occur even in the examples that have been taken from the econometric literature. We shall also see that one can always reduce the unbalanced case to the balanced case by introducing new variables. This will be discussed in detail in

Section 7, and in connection with the examples 3.4, 6.1 and 6.2.

Example 3.3. Consider the system

$$\Delta^2 \begin{bmatrix} x_{1t} \\ x_{2t} \\ x_{3t} \end{bmatrix} = \begin{bmatrix} 1 & \Delta & \Delta^2 \\ 1 & -\Delta & 2\Delta^2 \\ 0 & -2\Delta & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \epsilon_{3t} \end{bmatrix}, t \geq 0$$

If we express C(L) as a function of $\Delta$, we get

$$C(L) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \Delta \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \end{bmatrix} + \Delta^2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = C_0 + \Delta C_1 + \Delta^2 C_2.$$

It easily follows that

$$N_0 = sp\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \} \;;\; N_0 \cap N_1 = sp \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix};\; N_0 \cap N_1 \cap N_2 = \{0\}$$

and hence $m_0 = 2$, $m_1 = 1$, $m_2 = \ldots = 0$. Thus $k=2, n=3$ and since det $C(\Delta) = 2\Delta^3$ we have $r = n = 3$. The adjoint matrix becomes

$$\bar{C}(\Delta) = \begin{bmatrix} 4\Delta^3 & -2\Delta^3 & 3\Delta^3 \\ 0 & 0 & -\Delta^2 \\ -2\Delta & 2\Delta & -2\Delta \end{bmatrix}$$

$$= \Delta \left[ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 2 & -2 \end{bmatrix} + \Delta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} + \Delta^2 \begin{bmatrix} 4 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right]$$

$$= \Delta \left[ \bar{C}_1 + \Delta \bar{C}_2 + \Delta^2 \bar{C}_3 \right]$$

The equation now becomes

$$\Delta^3 [ \bar{C}_1 + \Delta \bar{C}_2 + \Delta^2 \bar{C}_3 ] x_t = 2\Delta^3 \epsilon_t.$$

Now cancel $\Delta^n = \Delta^3$ and find the term corresponding to $d = 2$ which gives the autoregressive part of the model. Then

$$\Delta^2 \bar{C}_3 x_t = -\bar{C}_1 x_t - \Delta \bar{C}_2 x_t + 2\epsilon_t + a + bt + ct^2$$

where a, b, and c are determined by the values before time $t = 0$.

Now insert the three matrices $\bar{C}_i$ and we get

$$\Delta^2\begin{bmatrix} 4x_{1t}-2x_{2t}+3x_{3t} \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 0 \\ x_{1t}-x_{2t}+x_{3t} \end{bmatrix} + \Delta\begin{bmatrix} 0 \\ x_{3t} \\ 0 \end{bmatrix} + 2\begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \epsilon_{3t} \end{bmatrix} + P_2(t)$$

or equivalently

$$\Delta^2[4x_{1t}-2x_{2t}+3x_{3t}] = 2\epsilon_{1t} + S_2(t) \tag{3.10}$$

$$2[x_{1t}-x_{2t}+x_{3t}] = -2\epsilon_{3t} + Q_2(t) \tag{3.11}$$

$$\Delta x_{3t} = -2\epsilon_{2t} + R_2(t). \tag{3.12}$$

Note that the equations (3.11) and (3.12) are special cases of (3.9) which express that certain linear combinations are integrated of lower order than 2. From (3.11) we see that $x_{1t}-x_{2t}+x_{3t}$ is stationary apart from the trend. Note also that $x_{1t}$ and $x_{2t}$ are integrated of order 2 and $x_{1t}-x_{2t}$ is integrated of order 1, but the variable that makes $x_{1t}-x_{2t}$ integrated of order 1 is the same as that in $-x_{3t}$, hence $x_{1t}-x_{2t}+x_{3t}$ is reduced to stationarity. Thus (3.11) is an example of a cointegration relation where all the components are not necessarily integrated of the same order. Note that the example has $n = r = 3$ and in this case the rows of $C_1$ span $M_1$ whereas the rows of $C_1$ and $C_2$ span $M_0$ , see Theorem 7.4.

Example 3.4. Let $x_t$ be given by the equations

$$\Delta\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} 1+\Delta & -\Delta \\ \Delta & 0 \end{bmatrix}\begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix}$$

$$C(L) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \Delta\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} ; \quad \bar{C}(L) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \Delta\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

and $\det C(L) = \Delta^2$. We find $N_0 = sp\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $N_1 = sp\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ , hence $M_0 = N_0$ and $M_1 = N_1$ , giving the indices $k = 1$, $m_0 = 1$, $m_1 = 0$ , and hence $n = 1$, whereas $r = 2$. Thus we are in the unbalanced case but $\bar{C}_n = \bar{C}_1 \neq 0$ and we can apply Theorem 3.1. In the equation

$$\bar{C}(L)\Delta x_t = \Delta^2\epsilon_t$$

we can cancel $\Delta^n = \Delta$ and the resulting equation is a general error correction

model. The autoregressive part is then given by the terms involving $\Delta x_t$. We then get

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \Delta x_t = -\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x_t + \Delta \epsilon_t$$

which shows that $x_{2t}$ is stationary and hence that (0,1) is a cointegration vector of order 1. The last equation is

$$\Delta(x_{2t}-x_{1t}) = -x_{2t} + \Delta\epsilon_{2t},$$

where the left hand side is the autoregressive part of the equation and on the rigth hand side $x_{2t}$ represents the error correction term and $\epsilon_{2t}$ the shocks. If we solve this for $x_{2t}-x_{1t}$ then the major contribution to the variance will be the term $x_{2t}$, since $V(\sum_{s=0}^{t} x_{2s})$ increases, whereas $V(\sum_{s=0}^{t} \Delta\epsilon_{2s})$ is constant in t. Thus the case $r > n$ can be interpreted as the case where the major contribution to the variance is given by the error corection terms, whereas the shocks only play a minor role. Another way of expressing this is that we have combined the major part of the shocks into an expression which we can give an interpretation, namely the error correction term. The case $r > n$, however, can be reduced to $r = n$ by transforming the variables into new variables which are linear combinations of variables of the same order of integration, namely

$$y_{1t} = x_{1t}$$

$$y_{2t} = -\Delta x_{1t} + x_{2t}.$$

The choice of these new variables is based on an analysis of the matrices $\{C_i\}$ such that the variables are linear combinations of suitably differenced components of $x_t$. This is discussed in more detail in Section 7.

The equation for the new variables is found by multiplying by the matrix

$$\begin{bmatrix} 1 & 0 \\ -\Delta & 1 \end{bmatrix}$$

and we find

$$\Delta\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\Delta & 1 \end{bmatrix}\begin{bmatrix} 1+\Delta & -\Delta \\ \Delta & 0 \end{bmatrix}\begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} = \begin{bmatrix} 1+\Delta & -\Delta \\ -\Delta^2 & \Delta^2 \end{bmatrix}\begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix}$$

The relevant matrix functions are now given by

$$C(L) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \Delta\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + \Delta^2\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\bar{C}(L) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \Delta\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \Delta^2\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

and det $C(L) = \Delta^2$. In this case $N_0 = sp\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $N_1 = sp\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $N_2 = sp\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and hence $M_0 = N_0$, $M_1 = M_0 = N_0 \cap N_1$ but $M_2 = \{0\}$. Thus $m_0 = 1$, $m_1 = 1$, $m_2 = 0$ which shows that $k = 2$ and $n = r = 2$. The error correction model looks like

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\Delta^{-1}y_t + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}y_t + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\Delta y_t = \epsilon_t.$$

The first two terms represent error corrections and show that $\Delta^{-1}y_{2t}$ and $y_{2t}$ are stationary, whereas $y_{1t}$ satisfies the autoregressive model

$$\Delta y_{1t} = \epsilon_{1t} - y_{2t}$$
$$\Delta y_{1t} = \epsilon_{2t} - y_{2t} - \Delta^{-1}y_{2t}.$$

These equations are clearly equivalent in view of the representation of $y_{2t}$ as a function of $\epsilon_t$. The first analysis of this example showed that $x_{2t}$ is in fact stationary. A fact which is obvious from the defining equation, since one can cancel a $\Delta$ in the equation for $x_{2t}$. It is probably a good idea to start out with variables which are integrated of the same order. The second analysis offers a more interesting point, since we find that not only is $y_{2t} = x_{2t} - \Delta x_{1t}$ stationary, but in fact integrated of order -1, which shows that $x_{1t} - \Delta^{-1}x_{2t}$ is stationary. We have thus found a cointegration relation between two variables of the same order, $x_{1t}$ and $\Delta^{-1}x_{2t}$.

## 4. The error correction model derived from the autoregressive representation

Let the process $\{x_t, t \geq 0\}$ be given by the equation

$$(4.1) \qquad A(L)x_t = \Delta^p \epsilon_t \ , \ t \geq 0,$$

where $A(z)$ is holomorphic for $|z| < 1+\rho$ and non-singular for $z \neq 1$ , but $A(1) \neq 0$. We define the coefficients $\{A_i\}$ by expanding $A(z)$ around $z = 1$, $A(z) = \sum_{i=0}^{\infty} (1-z)^i A_i$ , $|z| < \rho$. We want to interpret the equation (4.1) as an error correction model, see Definition 3.1. For this it turns out that we need to calculate the numbers $(k', r', n')$ for the transposed matrix function $A(z)'$.

Theorem 4.1. The process $x_t$ given by (4.1) is integrated of order less than or equal to $r'-n'+k'-p$ and in the expansion

$$(4.2) \quad \Delta^{r'-n'-p} A(L)x_t = \sum_{j=0}^{k'-1} \Delta^{r'-n'+j-p} A_j x_t + A_{k'}(L)\Delta^{r'-n'+k'-p} x_t = f(L)\Delta^{r'-n'} \epsilon_t$$

all terms are stationary. If either $\bar{A}_{n'-k'} \neq 0$ or $r' = n'$ then (4.2) is an error correction model of order $r'-n'+k'-p$.

Proof. It follows easily from the definition of the adjoint matrix, that $(\overline{A'(z)}) = (\overline{A(z)})'$ and hence that we have from (7.4)

$$\bar{A}(L) = \Delta^{n'-k'} \bar{A}_{n'-k'}(L)$$

and from (7.6)

$$(4.3) \qquad A_j \bar{A}_{n'-k'}(L) = \Delta^{k'-j} A_j \bar{A}_{n'-j}(L) \ , \ j = 1, \ldots, k.$$

Now multiply (4.1) by $\bar{A}(L)$, then we get

$$f(L)\Delta^{r'} x_t = \Delta^{n'-k'+p} \bar{A}_{n'-k'}(L)\epsilon_t$$

which shows that

$$(4.4) \qquad f(L)\Delta^{r'-n'+k'-p} x_t = \bar{A}_{n'-k'}(L)\epsilon_t$$

is stationary. If $\bar{A}_{n',-k'}(1) \neq 0$ then $x_t$ is integrated of order r'-n'+k'-p. If r' = n', then, by Theorem 7.4, the rows of $\bar{A}_{n',-k'}$ span $M'_{k',-1}$ which is non-empty and hence again $\bar{A}_{n',-k'} \neq 0$ such that $x_t$ is integrated of order k'-p which proves (3.4).

Now multiply (4.4) by $A_j$ and we get from (4.3) that the first terms of the expansion vanish, and that

$$\Delta^{r-n'+k'-p} f(L) A_j x_t = \Delta^{k'-j} A_j \bar{A}_{n',-j}(L) \epsilon_t$$

which shows that

$$\Delta^{r'-n'+j-p} A_j x_t \ , \ j = 1,\dots,k$$

is stationary which proves (3.3). Since $A_{k'} = A_{k'}(1) \neq 0$ , by the definition of k', it follows that we have an error correction model. In the case when r' > n' one may get $\bar{A}_{n',-k'} = 0$ in which case $x_t$ will be integrated of lower order and one may have to cancel some more powers of $\Delta$ before the model can be interpreted as an error correction model, but the condition that $\bar{A}_{n',-k'} \neq 0$ ensures that no power can be cancelled and that the results hold.

Example 4.1. Consider the equations

$$x_{1t} - x_{2t} + \Delta x_{2t} = \epsilon_{1t}$$

$$\Delta(x_{1t} - x_{2t}) = \epsilon_{2t}. \tag{4.5}$$

It is clear from the second equation, that $x_{1t}-x_{2t}$ is integrated of order 1 , and hence it follows from the first equation that $\Delta x_{2t}$ is integrated of order 1 and hence that $x_{2t}$ is of order 2. Thus all terms are not stationary, which means that (4.5) can not directly be interpreted as an error correction model in the sense of Definition 3.1. We shall give a formal analysis as follows: We find

$$A'(L) = \begin{bmatrix} 1 & \Delta \\ -1+\Delta & -\Delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + \Delta \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

giving $k' = 1, m_0' = 1, m_1' = 0$ and $n' = 1$, while $r' = 2$. The expansion of $\Delta^{r'-n'-d}A(L)x_t$ now reduces to multiplying through by $\Delta$ in the equation defining $x_t$, and we get

$$\begin{aligned} \Delta(x_{1t}-x_{2t}) + \Delta^2 x_{2t} &= \Delta\epsilon_{1t} \\ \Delta^2 x_{1t} - \Delta^2 x_{2t} &= \Delta\epsilon_{2t} \end{aligned}$$

Theorem 4.1 now gives the order of $x_t$ is less than or equal to $r'-n'+k'-p = 2$. Clearly $r' > n'$, but it is easy to see that $\bar{A}_{n',-k'} = \bar{A}_0 \neq 0$, which shows that the order in this case is equal to 2 and that (4.5) has to be multiplied by $\Delta$ to become an error correction model, and that the error correction term becomes $\Delta(x_{1t}-x_{2t})$. Thus in order to interprete the equation as an error correction model one first has to multiply by $\Delta$.

## 5. Granger causality.

We shall consider the special case of (3.1) where $x_t = (y_t, z_t)'$ and $C(L)$ and $\epsilon_t$ are partitioned accordingly

$$\Delta^d \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} P(L) & Q(L) \\ 0 & R(L) \end{bmatrix} \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix}$$

It is easily seen that $\det C(L) = \det P(L) \det R(L)$ and that

$$\bar{C}(L) = \begin{bmatrix} \bar{P}(L) \det R(L) & -\bar{P}(L)Q(L)\bar{R}(L) \\ & \\ 0 & \bar{R}(L) \det P(L) \end{bmatrix} .$$

Apart from the assumptions on $C(z)$ stated in section 3 we shall assume that $R(z)$ is non-singular and that $P(1) \neq 0$. Then one finds

$$R(L)^{-1}\Delta^d z_t = \epsilon_{2t}$$

and

$$\Delta^d(y_t - Q(L)R(L)^{-1}z_t) = P(L)\epsilon_{1t}. \tag{5.1}$$

One can define $\Pi(L) = Q(L)R(L)^{-1}$ and let $\Pi(1) = \Pi$ be the impact of $z_t$ on $y_t$. We shall call $\Pi(L)z_t$ the revealed target and $y_t - \Pi(L)z_t$ the target error, see

Kloek(1983). We let (k,n,r) denote the indices for the matrix function P(z), and get from Theorem 3.1, that if $r = n$ or $\bar{P}_n \neq 0$ then we have the error correction model

$$(5.2) \quad \sum_{j=1}^{k} \Delta^{d-j}\bar{P}_{n-j}(y_t - \Pi(L)z_t) + \bar{P}_n(L)\Delta^{d}(y_t - \Pi(L)z_t) = f(L)\Delta^{r-n}\epsilon_{1t}.$$

We shall interprete this equation as follows: From (5.1) it follows that the target error is integrated of order d, since $P(1) \neq 0$. Hence $y_t$ can be tracked by the target $\Pi(L)z_t$, such that the difference, the target error, becomes integrated of order d. The error correction terms in (5.1) signify that certain linear combinations of the variable $y_t$ can be tracked closer, in the sense that these linear combinations of the target errors are integrated of lower order, or in other words the target error is cointegrated.

Davidson(1983) compares the dynamic target $\Pi(L)z_t$, relevant for a steady-state growth world, with the static target $\Pi z_t$ relevant for a static equlibrium world. He then calls $\Pi(L)$ trend neutral of order m if $\Pi(L)t^j = \Pi t^j$ for $j = 0,1,\ldots,m$, and derives the restrictions to be placed on the structural parameters of the equation system for this to hold.

Let us next assume that the equations are given in the autoregressive form:

$$\begin{bmatrix} F(L) & G(L) \\ 0 & H(L) \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \Delta^p \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix}$$

We assume that H(z) is non-singular , and that $F(1) \neq 0$.

The first equation is

$$F(L)y_t + G(L)z_t = \Delta^p\epsilon_{1t}$$

which can be written

$$(5.3) \quad F(L)(y_t + \Delta^{-r'}f(L)^{-1}\bar{F}(L)G(L)z_t) = \Delta^p\epsilon_{1t}.$$

Here (k',n',r') are the indices for the matrix function F(L)'. Hence the

revealed target is

$$\Pi(L) = -\Delta^{-r'} f(L)^{-1}\bar{F}(L)G(L)z_t$$

where $\Pi(L)$ can be replaced by the first terms in the expansion

$$\Pi(L) = \sum_{j=-r'}^{p-1} \Delta^j \Pi_j + \Delta^p \Pi_p(L)$$

since $\Pi_p(L)\Delta^p z_t$ is stationary. Now in case $r' = n'$ or $\bar{F}_{n',-k'} \neq 0$ the equation (5.3) gives rise to an error correction model for the target error of the form

$$\sum_{j=0}^{k'-1} \Delta^{r'-n'+j-p} F_j[y_t - \Pi(L)z_t] + F_{k'}(L)[y_t - \Pi(L)z_t] = f(L)\Delta^{r'-n'}\epsilon_{1t}$$

of order $r'-n'+k'-p$. A simple example of this is given in Section 6.

**6. Examples.** Consider the model proposed by Hendry and von Ungern-Sternberg (1981) and discussed by Davidson (1983).

Example 6.1 Let $(c_t, y_t, l_t)$ denote the logarithm of consumption, disposable income, and personal sector liquid assets respectively. The model takes the form

$$(6.1)\qquad \Delta c_t = \beta\Delta y_t + \gamma_{11}(y_{t-1} - c_{t-1}) + \gamma_{12}(y_{t-1} - l_{t-1}) + \epsilon_{1t}$$

$$(6.2)\qquad \Delta l_t = \gamma_{21}(y_{t-1} - c_{t-1}) + \epsilon_{2t}.$$

To complete the system we shall add a third equation explaining how $y_t$ is generated by $\epsilon_{3t}$. Then the equations will have the form discussed in Section 5 , and we can solve for $c_t$ and $l_t$ in terms of $y_t$ and $\epsilon_t$. Let

$$f(L) = \Delta\gamma_{11} + \Delta^2(1-\gamma_{11}) - \gamma_{12}\gamma_{21}(1-\Delta)^2.$$

We shall assume that $f(z) \neq 0$ for $z \neq 1$. Then we find

$$\begin{bmatrix} \Delta + \gamma_{11}L & \gamma_{12}L \\ \gamma_{21}L & \Delta \end{bmatrix} \begin{bmatrix} c_t \\ l_t \end{bmatrix} = \begin{bmatrix} \beta\Delta + \gamma_{11}L + \gamma_{12}L \\ \gamma_{21}L \end{bmatrix} y_t + \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix}$$

and hence

$$(6.3)\quad \begin{bmatrix} c_t \\ l_t \end{bmatrix} = f^{-1}(L) \begin{bmatrix} \Delta & -\gamma_{12}L \\ -\gamma_{21}L & \Delta + \gamma_{11}L \end{bmatrix} \left[ \begin{bmatrix} \beta\Delta + \gamma_{11}L + \gamma_{12}L \\ \gamma_{21}L \end{bmatrix} y_t + \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} \right]$$

The first term on the rigth hand side is the target $\Pi(L)y_t$ and the equation (6.3) is the target relation. Now different models for $y_t$ will give different behaviour of $c_t$ and $l_t$. We shall consider two cases

Case 1:

$$\Delta y_t = g + \epsilon_{3t}$$

which specifies a random walk with drift for $y_t$, and

Case 2:

$$\Delta y_t = g + \Delta^{-1}\epsilon_{3t} = g + \sum_{s=0}^{t} \epsilon_{3s}.$$

both cases has $\Delta^2 y_t$ stationary with zero mean, and since a target relation is

given up to stationary terms we find that (6.3) reduces to

(6.4) $$c_t = y_t - \Delta y_t/\gamma_{21} + \text{stationary terms}$$

(6.5) $$l_t = y_t + \Delta y_t((\beta-1)\gamma_{21} + \gamma_{11})/\gamma_{21}\gamma_{12} + \text{stationary terms.}$$

If we now take expectations given the negative past we get

$$E(c_t) = E(y_t) - g/\gamma_{21}$$

$$E(l_t) = E(y_t) + ((\beta-1)\gamma_{21} + \gamma_{11})g/\gamma_{21}\gamma_{12}$$

which are the equations one would get by formally letting $\Delta y_t = g$ be non-stochastic and equating the $\epsilon$'s to zero, see Davidson(1983).

Thus both cases give rise to the same long-term relations, but it is seen that in case 1, we have that $\Delta y_t - g$ is stationary, and hence $y_t$ is integrated of order 1 and therefore the same holds for $c_t$ and $l_t$. Thus (6.1) and (6.2) form an error correction model with an autoregressive representation of $\Delta c_t$, $\Delta l_t$, and $\Delta y_t$, where $c_t$ and $l_t$ are cointegrated with $y_t$ such that $c_t-y_t$ and $l_t-y_t$ are stationary and enter the equations with suitable coefficients.

In case 2, however, $\Delta^2 y_t$ is stationary which implies that $c_t$ and $l_t$ are integrated of order 2. Thus by differencing (6.1) and (6.2) we get an autoregressive representation of $\Delta^2 c_t$ and $\Delta^2 l_t$. Note that in this case $\epsilon_{1t}$ and $\epsilon_{2t}$ enter only in the differenced form, which shows that the main contribution to the variance of $c_t$ and $l_t$ comes from $\Delta^2 y_t$ and the error correction terms $\Delta(l_t-y_t)$ and $\Delta(c_t-y_t)$. Thus the interpretation of (6.1) and (6.2) as an error correction model depends on the model specification for the exogeneous variable $y_t$.

We shall now show how the formal procedures developed in Sections 3-5 can be applied to this example. If we write the case 2 in the autoregressive form

(6.6) $$A(L)x_t = \epsilon_t$$

where $x_t' = (c_t, l_t, y_t)$ then

$$A(L) = \begin{bmatrix} \gamma_{11}+\Delta(1-\gamma_{11}) & \gamma_{12}(1-\Delta) & -\gamma_{11}-\gamma_{12}+\Delta(\gamma_{11}+\gamma_{12}-\beta) \\ \gamma_{21}(1-\Delta) & \Delta & -\gamma_{21}(1-\Delta) \\ 0 & 0 & \Delta^2 \end{bmatrix}$$

which by the analysis of Section 4 has $k' = 1$, $n' = 1$, and $r' = 2$, whereas $p = 0$. Hence Theorem 4.1 shows that we shall multiply (6.6) by $\Delta^{r'-n'-p} = \Delta$. It is easily seen that $\bar{A}_{n'-k'} = \bar{A}_0 \neq 0$, such that the resulting equation gives an error correction model of order 2, since $r'-n'+k'-p = 2$. We then obtain the information from the analysis that $\Delta(c_t-y_t)$ and $\Delta(l_t-y_t)$ are the stationary error correction terms. Note that $r' > n'$ such that we are in the unbalanced case.

One can reduce to the case $r' = n'$ by introducing new variables which are found by analysing $A(L)'$, see Theorem 7.5, and the comments at the end of Section 7. It turns out that the new variables, in which the problem becomes balanced are

$$u_{1t} = c_t - y_t + \Delta y_t/\gamma_{21}$$
$$u_{2t} = l_t - y_t + \Delta y_t(\gamma_{21}(1-\beta)-\gamma_{11})/\gamma_{12}\gamma_{21}$$
$$u_{3t} = y_t$$

In terms of the new variables the equations now become

$$\gamma_{11}u_{1t} + \gamma_{12}u_{2t} + (1-\gamma_{11})\Delta u_{1t} - \gamma_{12}\Delta u_{2t} - (1-\beta-\gamma_{21}^{-1})\Delta^2 u_{3t} = \epsilon_{1t}$$
$$\gamma_{21}u_{1t} - \gamma_{21}\Delta u_{1t} + \Delta u_{2t} + (1 + (\beta - 1 + \gamma_{11}\gamma_{21}^{-1})\gamma_{12}^{-1})\Delta^2 u_{3t} = \epsilon_{2t}$$
$$\Delta^2 u_{3t} = \epsilon_{3t}$$

and hence we get an error correction model with an autoregressive model for $\Delta^2 y_t$ explained by the error correction terms which are recognised as the target errors (6.4) and (6.5).

If we modify the equation (6.1) to give an integral correction equation we get

(6.7) $$\Delta c_t = \beta\Delta y_t + \gamma_{11}(y_{t-1}-c_{t-1}) + \partial_{11}\Delta^{-1}(y_{t-1}-c_{t-1}) + \epsilon_{1t}$$

which together with (6.2) determines a system of equations for $(c_t, l_t)$. We solve (6.7) in the form

$$c_t = y_t + (\Delta^2 y_t(\beta-1) + \Delta\epsilon_{1t})/(\vartheta_{11} + \Delta(\gamma_{11}-\vartheta_{11}) + \Delta^2(1-\gamma_{11})).$$

This shows that if $\Delta^2 y_t$ is stationary, then $c_t - y_t$ is stationary, and hence $c_t$ is integrated of order 2, and the target relation is

$$c_t = y_t + \text{stationary terms.}$$

The similar relation for $l_t$ becomes

$$l_t = (1-\beta)\gamma_{21}\Delta y_t \vartheta_{11}^{-1} + \Delta^{-1}\epsilon_{2t} + \text{stationary terms.}$$

The formal analysis proceeds as follows: We multiply through by $\Delta$ to avoid the negative power. We then have an expression of the form (4.1) with d = 1 and

$$A(L) = \begin{bmatrix} \vartheta_{11}+\Delta(\gamma_{11}-\vartheta_{11})+\Delta^2(1-\gamma_{11}) & 0 & -\vartheta_{11}+\Delta(\vartheta_{11}-\gamma_{11})+\Delta^2(\gamma_{11}-\beta) \\ \gamma_{21}\Delta(1-\Delta) & \Delta^2 & -\gamma_{21}\Delta(1-\Delta) \\ 0 & 0 & \Delta^3 \end{bmatrix}$$

We find k' = 2, n' = 4, and r' = 5, but some calculation shows that $\bar{A}_{n',-k'} = \bar{A}_2 \neq 0$. Hence the process $(c_t, l_t, y_t)$ is of order $r'-n'+k'-p = 2$, and we shall multiply through by $\Delta^{r'-n'-d'} = \Delta^0$ in order that the equation can be interpreted as an error correction model. We thus find that $c_t - y_t$ is a stationary error correction term which appears with coefficient $-\vartheta_{11}$ in the expression for $\Delta^2 c_t$ and in the form $-\Delta\gamma_{21}(c_t - y_t)$ in the equation for $\Delta^2 l_t$.

Since r' > n' we can introduce a new variable, and an investigation of the columns of A(L), or the rows of A(L)', will show that the new variable is

$$u_t = c_t - y_t - \Delta^2 y_t(\beta-1)/\vartheta_{11}$$

together with $l_t$ and $y_t$. In the new variables the matrix A becomes

$$A(L) = \begin{bmatrix} \vartheta_{11}+\Delta(\gamma_{11}-\vartheta_{11})+\Delta^2(1-\gamma_{11}) & 0 & \Delta^3[\gamma_{11}-\vartheta_{11}+\Delta(1-\gamma_{11})](\beta-1)/\vartheta_{11} \\ \gamma_{21}\Delta(1-\Delta) & \Delta^2 & \gamma_{21}\Delta^3(1-\Delta)(\beta-1)/\vartheta_{11} \\ 0 & 0 & \Delta^3 \end{bmatrix}$$

which is seen to have $k' = 3$, $n' = r' = 5$. Hence the process $(u_t, l_t, y_t)$ is integrated of order $r'-n'+k'-d = 2$, and we shall multiply through by $\Delta^{r'-n'-d} = \Delta^{-1}$. Thus we shall cancel a factor $\Delta$ again and we find that the error correction terms now are $\Delta^{-1}u_t, u_t, \Delta u_t$, and $\Delta l_t$ which appear in the autoregressive model for $\Delta^2 y_t$. Thus we find that $u_t = c_t - y_t - \Delta^2 y_t(\beta-1)/\partial_{11}$ is in fact integrated of order $-1$, which means that $c_t$ can be tracked extremely well by $v_t + \Delta^2 y_t(\beta-1)/\partial_{11}$, in the sense that the error does not accumulate, i.e. the sum of the target errors $\sum_{s=0}^{t} u_s$ has a bounded variance.

The next example we shall consider is a model proposed by Harvey(1982) for a stochastically varying trend.

Example 6.2. Let the variables $y_t, m_t, \beta_t$, and $x_t$ be given by the equations

$$y_t = m_t + \alpha x_t + \epsilon_{1t}$$

$$\Delta m_t = \beta_t + \epsilon_{2t}$$

$$\Delta \beta_t = \epsilon_{3t}$$

$$\Delta x_t = \epsilon_{4t}.$$

This is an autoregressive model, and we find

$$A(L) = \begin{bmatrix} 1 & -1 & 0 & -\alpha \\ 0 & \Delta & -1 & 0 \\ 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & \Delta \end{bmatrix}$$

which has determinant $\det A(L) = \Delta^3$, and hence $r' = 3$, and we find $k' = 1$ and $n' = 2$. Thus we are in the unbalanced case, but it is easily seen, that $\bar{A}_{n'-k'} = \bar{A}_1 \neq 0$. Hence Theorem 4.1 shows that the process is integrated of order $r'-n'+k'-p = 2$, and that a factor of $\Delta^{r'-n'-p} = \Delta$ is missing in the equations before they can be interpreted as an error correction model. Then the term involving $\Delta^2$ will be the autoregressive part and those with $\Delta$ the

error correction terms. It then follows that $\Delta(y_t - m_t - \alpha x_t)$ and $\Delta\beta_t$ are the stationary error correction terms, and that the only error correction equation where anything is corrected is

$$\Delta^2 m_t = \Delta\beta_t + \Delta\epsilon_{2t}.$$

It is seen that the main contribution to the variance of $m_t$ is due to the stationary error correction term $\Delta\beta_t$. Note that $y_t$ and $m_t$ are integrated of order 2, and that $y_t - m_t$ is integrated of order 1, whereas $y_t - m_t - \alpha x_t$ is integrated of order 0.. Hence we have an example where three variables are involved which are not of the same order.

Since r' > n' a differencing was needed. It can be avoided by introducing the new variable $u_t = \Delta m_t - \beta_t$. The problem is now balanced, and the equations become

$$\begin{aligned} y_t - m_t - \alpha x_t &= \epsilon_{1t} \\ u_t &= \epsilon_{2t} \\ \Delta^2 m_t - \Delta u_t &= \epsilon_{3t} \\ \Delta x_t &= \epsilon_{4t} \end{aligned}$$

In this case k' = 2, n' = r' = 3, and the process is of order r'-n'+k'-p = 2. Hence the imbalance in A has been removed by the change of variables and the new equation can be viewed as an error correction model ,where we now get the information that $y_t - m_t - \alpha x_t$ and $u_t$ are the stationary error correction terms, whereas $x_t$ is the error correction term of order 1. The relevant error correction model now becomes

$$\Delta^2 m_t = \Delta u_t + \epsilon_{3t}$$

where now the error correction term contributes less than the shocks to the variation of $y_t$.

## 7. Mathematical results

Consider a matrix valued function $C(z)$ which is defined in an open disc $D = \{z \; ; \; |z| < 1+\rho \}$ in the complex plane. The function is called holomorphic if the n'th derivative exists for all n, and it is a well known result that the Taylor series expansion of a holomorphic function at a point $z \in D$ converges in the largest open disc contained in D, see for instance Thron(1953).

We shall investigate the function $C(z)$ around the point $z = 1$, and we assume that the matrix $C(z)$ is non-singular for $z \neq 1$, and that $C(1)$ is singular, but $\neq 0$. We define the coefficients $\{C_j \; ; \; j = 0,1,\ldots \}$ by the expansion

$$C(z) = \sum_{j=0}^{\infty} (1-z)^j C_j \; , \quad |1-z| < \rho$$

We shall repeatedly use the fact that if $C_n(z)$ is defined by

$$C(z) = \sum_{j=0}^{n-1} (1-z)^j C_j + (1-z)^n C_n(z)$$

then $C_n(z)$ is a holomorphic function in D. This follows since the functions $C(z)$, $\sum_{j=0}^{n-1} (1-z)^j C_j$, and $(1-z)^{-n}$ are holomorphic in D as long as $z \neq 1$. At the point 1, however, the function $C_n(z)$ has the expansion $C_n(z) = \sum_{j=0}^{\infty} (1-z)^j C_{j+n}$ $|z-1| < \rho$, which shows that $C_n(z)$ is also holomorphic at the point 1. We want to give a representation for the determinant of $C(z)$ and for $\bar{C}(z)$ the adjoint matrix defined by

$$\bar{C}_{ij}(z) = (-1)^{i+j} \det C^{ji}(z)$$

where $C^{ji}(z)$ is obtained from $C(z)$ by deleting row j and column i. Note that $\bar{C}(z)$ is also holomorphic since each element is given as a finite sum of finite products of holomorphic functions.

We define the coefficients $\{\bar{C}_j\ ;\ j = 0,1,\dots\}$ by the expansion at $z = 1$ $\bar{C}(z) = \sum_{j=0}^{\infty} (1-z)^j \bar{C}_j$. We define the null spaces

$$N_j = \{ x \in R^m \mid x'C_j = 0\},\ j = 0,1,\dots$$

and

$$M_j = N_0 \cap N_1 \cap \dots \cap N_j.$$

Then, since $C(z)$ is regular, there is no vector $x$ which makes all $C_j$ zero, hence $M_j = \{0\}$ for $j \geq k$, say. Note that $C_k$ must be non-zero, and that $M_{k-1} \neq \{0\}$, and that $C_0 = C(1) \neq 0$ implies that $M_0 \neq R^m$. We shall now define the index $n = \sum_{j=0}^{\infty} m_j$, where $m_j$ is the dimension of $M_j$. The basic idea is that if $x \in M_j$, then

$$x'C(z) = (1-z)^{j+1}\ x'C_{j+1}(z),$$

since the first $j+1$ terms $x'C_0,\dots,x'C_j$ are zero. This corresponds to the idea that $x$ is a cointegration factor of order $j+1$, and we shall use this to evaluate the determinant.

Theorem 7.1. The multiplicity $r$ of the root $z = 1$ of $\det C(z)$ is greater than or equal to $n$, hence there exists a function $f(z) \neq 0$, such that

$$\det C(z) = (1-z)^r f(z).$$

Proof. We want to choose a convinient coordinate system to evaluate the determinant and this is done as follows: From the relation

$$R^m \supset M_0 \supset M_1 \supset \dots \supset M_k = \{0\}$$

we get an orthogonal decomposition of $R^m$

$$R^m = V_0 + V_1 + \dots V_k,$$

where

$$V_j = M_{j-1} \cap M_j^{\perp} = N_0 \cap \dots \cap N_{j-1} \cap N_j^{\perp}$$

is of dimension $m_{j-1}-m_j$. Note that $C(1) \neq 0$ implies that $V_0 \neq \{0\}$ and that

the definition of k implies that $V_k \neq \{0\}$.

We define $\tilde{C}(z)$ by $x'\tilde{C}(z) = x'C_j(z)$, for $x \in V_j$, then

$$x'C(z) = (1-z)^j x'\tilde{C}(z) \ , \ x \in V_j \tag{7.1}$$

where $x'\tilde{C}(1) \neq 0$, since $x \notin M_j$. Thus $V_j$ is the space of cointegration factors of order j.

Now choose a basis $\{v_j;\ j = 1,\ldots,m\}$ for $R^m$, such that the vectors given by $\{v_j;\ j = m-m_{i-1}+1,\ldots,m-m_i\}$ span $V_i$. We use the notation $m_{-1} = m$. We define the order of $v_j$ to be $i(j)$, thus $i(j) = i$ if $v_j \in V_i$. Note that $\max_{1\leq j\leq m} i(j) = k$ and that $\sum_{j=1}^{m} i(j) = \sum_{i=0}^{\infty} i(m_{i-1}-m_i) = \sum_{i=0}^{\infty} m_i = n$. From (7.1) we find

$$v_p'C(z)v_q = C_{pq}(z) = (1-z)^{i(p)}\tilde{C}_{pq}(z)$$

or

$$C(z) = \text{diag}\{\ (1-z)^{i(p)}\ ;p = 1,\ldots,m\}\ \tilde{C}(z)$$

and hence that $\det C(z) = (1-z)^n \det \tilde{C}(z)$. Now let r be the multiplicity of the root $z = 1$ of $\det C(z)$, then $r \geq n$, since $C(1)$ may be singular. This completes the proof of Theorem 7.1.

<u>Corollary 7.2</u> For the adjoint matrix we have the result

$$\bar{C}_{ij}(z) = (-1)^{i+j}\det C^{ji}(z) = (-1)^{i+j}(1-z)^{r_{ji}} f_{ji}(z) \tag{7.2}$$

where $f_{ji}(z) \neq 0$ and $r_{ji} \geq n-p$ when $v_j \in V_p$.

<u>Proof</u> By deleting the row j with $v_j \in V_p$ from $C(z)$ we leave out a factor $(1-z)^p$ in the determinant and the index n is reduced by p.

The next result gives a representation of the matrix $\bar{C}(z) = \sum_{j=0}^{\infty} (1-z)^j \bar{C}_j$

<u>Theorem 7.3</u> The coefficients of the adjoint matrix satisfies

$$\bar{C}_j = 0 \ , \ j = 0,1,\ldots,n-k-1 \tag{7.3}$$

and hence

(7.4) $$\bar{C}(z) = (1-z)^{n-k}\,\bar{C}_{n-k}(z)$$

Further

(7.5) $$\bar{C}_{n-j}C_i = 0\ ,\ 0 \le i < j \le k$$

which shows that

(7.6) $$\bar{C}_{n-j}C(z) = (1-z)^j\bar{C}_{n-j}C_j(z)\ ,\ j = 1,\dots,k.$$

Proof Let us consider the coordinate system $\{v_j;\ j = 1,\dots,m\}$ from the proof of Theorem 7.1, and express $\bar{C}(z)$ in these coordinates. From (7.2) we find that $r_{ji} \ge n-p$ when $v_j \in V_p$, but we have $n-p \ge n-k$. Thus the smallest power that can occur in the expansion of $\bar{C}(z)$ is $n-k$, which shows that $\bar{C}_j = 0$, $j < n-k$, and this proves (7.3) and (7.4). To prove (7.5) and (7.6) we will show that

$$(\bar{C}_{n-j})_{pq}(C_i)_{qr} = 0 \text{ for all } p,q,r \text{ and } 0 \le i < j \le k.$$

We then get (7.5) by summing over q. Now if $q \le m-m_{j-1}$, then $v_q \in V_0 + \dots + V_{j-1}$ and $r_{qp} \ge n-(j-1)$. Thus the smallest power in the expression for $\bar{C}(z)_{pq}$ is $(1-z)^{n-(j-1)}$ which shows that $(\bar{C}_{n-j})_{pq} = 0$.

Similarly if $q > m-m_i$, i.e $v_q \in V_{i+1} + \dots + V_k = M_i$, then clearly $v_q' C_i = 0$, and hence $(C_i)_{qr} = 0$. Now if we take $i < j$, then $m-m_i \le m-m_{j-1}$, which shows that all q values were considered, and this completes the proof of the relation (7.5) and (7.6).

The relation (7.5) shows that the rows of $\bar{C}_{n-j}$ are contained in the null spaces of $C_i$ whenever $i < j$, and in particular that the rows of

$$\bar{C}_{n-(i+1)},\dots,\bar{C}_{n-k}$$

are contained in $M_i = N_0 \cap \dots \cap N_i$. We can now prove

**Theorem 7.4.** If $r = n$ then $\sum_{j=0}^{k} \bar{C}_{n-j}C_j$ is proportional to the identity and the rows of $\bar{C}_{n-i-1},\ldots,\bar{C}_{n-k}$ span $M_i$ , $i = 0,1,\ldots,k-1$.

Proof. If $r = n$ then the matrix $\tilde{C}(1)$ is regular, see the proof of Theorem 7.1. From the relation

$$\bar{C}(z)C(z) = \det C(z)\ I_{m\times m}$$

we find from the fact that $\bar{C}_0 = \ldots = \bar{C}_{n-k-1} = 0$ and $\bar{C}_{n-j}C_i = 0$ for $0 \le i < j \le k$, that the first possibly non-zero term on the left hand side is

$$(1-z)^n \sum_{j=0}^{k} \bar{C}_{n-j}C_j$$

and if $\det C(z) = (1-z)^n f(z)$, $f(z) \ne 0$, then $\sum_{j=0}^{k} \bar{C}_{n-j}C_j$ is proportional to the identity and hence has rank m.

Now consider the terms $(\bar{C}_{n-j})_{pq}(C_j)_{qr}$. From the proof of Theorem 7.3 it follows that

$$(\bar{C}_{n-j})_{pq} = 0 \text{ for } q \le m-m_{j-1}$$

and that

$$(C_j)_{qr} = 0 \text{ for } q > m-m_j$$

hence

$$(\bar{C}_{n-j}C_j)_{pr} = \sum (\bar{C}_{n-j})_{pq}(C_j)_{qr}$$

where the summation is for q such that $m-m_{j-1} < q \le m-m_j$ or $v_q \in V_j$. This shows that the rank of $\bar{C}_{n-j}C_j$ is less than or equal to $m_{j-1}-m_j$. We then evaluate as follows

$$m = \operatorname{rank}\left(\sum_{j=0}^{k} \bar{C}_{n-j}C_j\right) \le \sum_{j=0}^{k} \operatorname{rank}(\bar{C}_{n-j}C_j) \le \sum_{j=0}^{k} (m_{j-1}-m_j) = m.$$

It follows that equality holds throughout and that

$$\text{rank}\bar{C}_{n-j} \geq \text{rank}(\bar{C}_{n-j}C_j) = m_{j-1}-m_j.$$

This completes the proof, since then

$$m_i \geq \text{rank}(\bar{C}_{n-i-1},\ldots,\bar{C}_{n-k}) \geq \sum_{j=i+1}^{k} (m_{j-1}-m_j) = m_i$$

which shows that the matrices on the left span all of $M_i$.

If $r > n$ we do not get so complete information, but we shall show below how the case $r > n$, the unbalanced case, can be reduced to the case $r = n$, the balanced case. The idea is that instead of taking only linear combinations of the x's we allow powers of $\Delta$ in the coefficients. We can then prove, that by transforming the variables we can increase n while keeping r fixeu, and we thus reduce to the balanced case after at most r-n transformations. A similar idea is the starting point for the work of Yoo(1985). If $r > n$ then $\tilde{C}(1)$ is singular and we can find a vector $a = \sum_{p=1}^{m} a_p v_p \in R^m$ such that $a'\tilde{C}(0) = 0$. Let s be the largest j for which $a_j \neq 0$, and define $T_a(z)$ by

$$v_j'T_a(z) = \begin{cases} v_j' & , j \neq s \\ a_s^{-1} \sum_{p=1}^{s} (1-z)^{i(s)-i(p)} a_p v_p' & , j = s . \end{cases}$$

We can then prove

Theorem 7.5 The matrix function

$$C_a(z) = T_a(z)C(z)$$

has indices $(r_a,n_a,k_a)$, where $r_a = r$, $n_a \geq n + 1$, and $k_a \geq k$.

Proof. Since det $T_a(z) = 1$ we clearly have $r_a = r$. For $j \neq s$, we have $v_j'C_a(z) = v_j'C(z)$, and it follows that the order of $v_j$ is the same for $C_a(z)$ as for C(s). If $j = s$, then

$$\begin{aligned} a_s v_s' C_a(z) &= \sum_{p=1}^{s} (1-z)^{i(s)-i(p)} a_p v_p' C(z) \\ &= \sum_{p=1}^{s} (1-z)^{i(s)-i(p)} a_p (1-z)^{i(p)} v_p' \tilde{C}(z) \end{aligned}$$

$$= (1-z)^{i(s)} a'\tilde{C}(z).$$

Hence, since $a'\tilde{C}(1) = 0$, we get that the order of $v_s$ is $\geq i(s) + 1$, which implies that $k_a$ is at least as large as $k$, whereas $n_a$ is greater than $n$.

Let us briefly discuss the relevance of the above formulation for the theory of time series. The reason that the holomorphic functions play a role is that if $\{z_t; -\infty < t < \infty\}$ is stationary, and if $B(z) = \sum_{i=0}^{\infty} z^i B_i^*$ is holomorphic for $|z| < 1+\rho$ then the process $y_t = B(L)z_t = \sum_{i=0}^{\infty} B_i^* z_{t-i}$ is a stationary process. The coefficients $B_i^*$ decrease exponentially fast in $i$, which shows that the process $\{y_t\}$ is well defined and it is easy to see that it is stationary.

We have throughout considered the matrices as linear transformations of the row vectors, using the notation $v'C(z)$. This comes from the fact that in the moving average model (3.1) the change of variable $y_t = Tx_t$ gives the relation

$$\Delta^d y_t = TC(z)\epsilon_t.$$

Thus by choosing a suitable $T$ we can change the variables to find a convinient coordinate system in which to calculate the determinant or, in case $T$ depends on $L$, to reduce the unbalanced case to the balanced case.

If the starting point of the investigation is the autoregressive model (4.1) then the change of variable $y_t = Tx_t$ implies the equation

$$A(L)T^{-1}y_t = [T'^{-1}A(L)']'y_t = \Delta^d \epsilon_t.$$

Thus the inverse of $T'$ operates on $A(L)'$. Now it is easy to see that from the definition of the adjoint, it follows that $(\overline{A(L)'}) = (\overline{A(L)})'$, and hence that the results of the previous Theorems can be aplied without problems.

Let us end this section by giving explicitly the transformation of the

variables that increases n by at least 1.

If

$$\Delta^d x_t = C(L)\epsilon_t$$

and $a'\tilde{C}(1) = 0$, then we can choose $y_t = T_a(L)x_t$ as follows

$$y_{jt} = \begin{cases} x_{jt} & , j \neq s \\ a_s^{-1} \sum_{p=1}^{s} \Delta^{i(s)-i(p)} a_p x_{pt} & , j = s . \end{cases}$$

If

$$A(L)x_t = \Delta^d \epsilon_t$$

and $a'\tilde{A}(1)' = 0$, then one can introduce the variable $y_t$ by

$$y_{jt} = \begin{cases} x_{jt} - a_s^{-1} \Delta^{i(s)-i(j)} a_j x_{st} & j < s \\ x_{jt} & j \geq s. \end{cases}$$

Examples of this are given in Section 6.

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A general error correction model is defined, which encompasses the usual error correction model as well as the integral correction model. It is shown that the model applies to autoregressive and moving average models, and the theory is applied to some examples receiving previous attention in the literature.

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